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STREAM FUNCTION FINITE ELEMENT SOLUTION OF STOKES FLOW WITH PENALTIES

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SUMMARY

A finite element method for solution of the stream function formulation of Stokes flow is developed. The method involves complete cubic non-conforming (C^0) triangular Hermite elements. This element fails the patch test. To correct the element and produce a convergent method we employ a penalty method to weakly enforce the desired continuity constraint on the normal derivative across the inter-element boundaries. Successful use of the method is demonstrated to require reduced integration of the inter-element penalty with a 1-point Gauss rule. Error estimates relate the optimal choice of penalty parameter to mesh size and are corroborated by numerical convergence studies. The need for reduced integration is interpreted using rank relations for an associated hybrid method.

KEY WORDS Stream Function Viscous Flow Non-conforming Elements Interface Penalties Reduced Integration

INTRODUCTION

There are three main formulations for Stokes and Navier–Stokes flows: (i) the primitive-variable or pressure–velocity formulation; (ii) the stream-function: vorticity formulation and (iii) the stream-function formulation.

The primitive-variable formulation and variants of this form have received considerable study using finite difference and more recently finite element methods. The problem requires solution of a system of second-order differential equations and the corresponding variational problem implies that C^0 elements will be conforming. The stream-function:vorticity form of the equations is also a second-order system and again C^0 elements suffice in the variational finite element method. There are, however, added difficulties associated with the boundary conditions for vorticity. Finally, the stream-function form results from substituting for the vorticity in the stream-function:vorticity equations. This yields the single biharmonic equation for the stream function in the case of stationary Stokes flow and a non-linear fourth-order equation for stationary Navier–Stokes problems.

In the present study we shall consider only the finite element variational formulation of the stream-function form of Stokes flow. As noted above the problem is of fourth order. This implies that the variational problem involves admissible functions whose second derivatives are to be square integrable. It suffices then that the global finite element basis have continuous derivatives for conformity. Several conforming finite elements have been developed for plate bending applications and are equally applicable here. Some examples of C^1 elements are the rectangle with tensor product Hermite basis of Bogner *et al.*,[1] the quintic triangle of Cowper *et al.*[2] and the reduced quintic of Bell[3] and Argyris *et al.*[4] Composite elements with interior basis constraints

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such as the triangle of Clough and Tocher[5] and non-conforming elements that satisfy the patch test[6] may also be used.

It is nevertheless interesting to note that apparently few studies have been made using C^1 triangles or other elements for solving the stream function equation. The most notable exception is the work by Olson and Tuann[7] using the C^1 quintic triangle.

It is well known that the C^1 triangle with complete polynomial basis necessitates the use of a high degree element (quintic with 21 degrees of freedom) and hence leads to a large element matrix (21×21) which adversely affects the bandwidth. For problems with smooth solutions, however, it is also clear that this element will produce highly accurate results for calculations with relatively coarse meshes. Given the variational statement and the weak condition on second derivatives, there are many practical problems which will not exhibit the high degree of regularity necessary to exploit the capability of the quintic element. The familiar driven cavity example and flow across a step are examples where singularities in the data influence the regularity of the solution and optimal rates will not be achieved.

Other techniques for enforcing C^1 continuity using Lagrange multipliers have been proposed[8–10] again in the context of plate bending. Recently we demonstrated that there may be a subtle linear dependence of the constraints when multipliers are used to this end and showed further that a penalty method may be formulated which does not exhibit this problem.[11]

In the present study we consider the use of the complete Hermite cubic triangle for the variational formulation of the stream function (biharmonic) equation. The element is only C^0 and the continuity of normal derivative across the interface between adjacent elements is enforced using a continuous penalty constraint term. Numerical experiments indicate that the method is practical if 1-point reduced Gauss quadrature is used for the penalty functional. Theoretical estimates and further numerical results are given.

STREAM-FUNCTION FORMULATION

The stream-function form of the stationary Navier–Stokes equation is

$$-\nu\Delta^2\psi + \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y = f \tag{1}$$

where ν is the viscosity and f the applied body force. For low Reynolds number flows the equation can be linearized to Stokes flow and we have the biharmonic operator with

$$\Delta^2\psi = -f/\nu \tag{2}$$

The weak variational forms of (1) and (2) are easily formulated from a weighted-residual approach. For the Navier–Stokes problem (1) we have: find $\psi \in H^2$ (the space of functions with square integrable second partial derivatives) satisfying the essential boundary conditions and such that

$$\int_\Omega \nu\Delta\psi\Delta w \,dx\,dy - \int_\Omega \Delta\psi[(\psi_y w)_x - (\psi_x w)_y]\,dx\,dy = \int_\Omega f w\,dx\,dy \tag{3}$$

for all admissible test functions w with $w = 0$ and $\partial w/\partial n = 0$ on that part of the boundary $\partial\Omega$ where essential conditions apply.

For Stokes flow the statement (3) simplifies to

$$\int_\Omega \nu\Delta\psi\Delta w\,dx\,dy = \int_\Omega f w\,dx\,dy \tag{4}$$

In our study it suffices that we consider the linear Stokes flow. The main results concerning the penalty formulation apply to the non-linear problem although the error estimates would have to be generalized to this case.

APPROXIMATE PROBLEM

Consider a bounded flow domain Ω discretized as a union Ω_h of triangles Ω_e Let $\{\phi_i\}$ be the global finite element basis defining the approximation space H^h on Ω_h. The approximate statement of the Stokes problem (4) is: find $\psi_h \in H^h$ satisfying the essential conditions on $\partial\Omega_h$ and such that

$$\int_{\Omega_h} \nu \Delta\psi_h \Delta w_h \,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega_h} f w_h \,\mathrm{d}x\,\mathrm{d}y \tag{5}$$

for all admissible $w_h \in H^h$. (A similar statement holds for the non-linear problem.)

In this analysis we are particularly interested in non-conforming elements, i.e. H^h is not a subspace of the solution space. This implies that the sequence of approximate stream-function solutions may not converge on mesh refinement to the true solution. To enforce the continuity of the normal derivative across the interface Γ_S between adjacent elements we add the restriction that the minimization problem giving (5) as its first variation should hold on a subset C satisfying the constraint

$$\left[\!\left[\frac{\partial\psi_h}{\partial n} \right]\!\right] = 0, \quad \text{on } \Gamma_S \tag{6}$$

where $[\![\cdot]\!]$ denotes the interface jump and Γ_S is any interface in the discretization. The penalized minimization problem corresponding to (5) and (6) is: find $\psi_h^\varepsilon \in H^h$ satisfying the essential boundary conditions and minimizing, for penalty parameter $0 < \varepsilon \ll 1$,

$$J_\varepsilon(v_h) = \sum_{e=1}^{E} \int_{\Omega_e} \left[\frac{\nu}{2}(\Delta v_h)^2 - f v_h \right] \mathrm{d}x\,\mathrm{d}y + \sum_{s=1}^{S} \frac{1}{2\varepsilon} \int_{\Gamma_S} \left[\!\left[\frac{\partial v_h}{\partial n} \right]\!\right]^2 \mathrm{d}s \tag{7}$$

where $s = 1, 2, \ldots, S$ is the number of element sides in the interior and e is the element index. Setting $\delta J_\varepsilon = 0$ at ψ_h^ε we obtain

$$\begin{aligned} &\sum_{e=1}^{E} \int_{\Omega_e} \nu \Delta\psi_h^\varepsilon \Delta w_h \,\mathrm{d}x\,\mathrm{d}y + \sum_{s=1}^{S} \frac{1}{\varepsilon} \int_{\Gamma_S} \left[\!\left[\frac{\partial\psi_h^\varepsilon}{\partial n} \right]\!\right] \left[\!\left[\frac{\partial w_h}{\partial n} \right]\!\right] \mathrm{d}s \\ &\quad = \sum_{e=1}^{E} \int_{\Omega_e} f w_h \,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{8}$$

for all admissible w_h.

Later we shall find it advantageous to use reduced integration for the penalty term. Let $I_s(g) = \int g\,\mathrm{d}s$ denote the n-point numerical Gauss quadrature on Γ_S. Then (8) becomes

$$\begin{aligned} &\sum_{e=1}^{E} \int_{\Omega_e} \nu \Delta\psi_h^\varepsilon \Delta w_h \,\mathrm{d}x\,\mathrm{d}y + \sum_{s=1}^{S} \frac{1}{\varepsilon} I_s\left(\left[\!\left[\frac{\partial\psi_h^\varepsilon}{\partial n} \right]\!\right] \left[\!\left[\frac{\partial w_h}{\partial n} \right]\!\right] \right) \\ &\quad = \sum_{e=1}^{E} \int_{\Omega_e} f w_h \,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{9}$$

To determine the dependence of ε on h we use $\varepsilon = Ch^\sigma$ with constant C and σ to be determined from the error estimates.

Introducing the element basis in (9), we determine the penalized finite element system for a given ε

and linear algebraic solution yields the penalty solution ψ_h^ε. For ε sufficiently small we anticipate $\psi_h^\varepsilon \to \psi_h$ where $\psi_h \to \psi$ on mesh refinement. That is, the penalty term appropriately corrects the non-conforming element. In practice we assemble the interface penalty matrix contributions in a separate loop over the interfaces.

ESTIMATES

We first note that the bilinear form on the left in (9) can be used to determine an appropriate norm $|||\cdot|||_h$ on H^h,

$$|||v|||_h = \sum_{e=1}^{E} \int_{\Omega_e} \nu(\Delta v)^2 \mathrm{d}x\mathrm{d}y + \sum_{s=1}^{S} \frac{1}{2\varepsilon} I_s\left(\left[\frac{\partial v}{\partial n}\right]^2\right) \tag{10}$$

Next we introduce the complete Hermite triangle (cubic with function value and first partials at the vertices and function at the centroid as degrees of freedom). Later we shall show using rank conditions that the method with 1-point integration is stable whereas the 2-point and 3-point schemes are not.

Theorem

Let Ω be a polygonal domain and let $f \in L^2(\Omega)$ such that $\psi \in H^r(\Omega)$, $r \geq 5/2$ is the solution of the variational Stokes problem. The finite element solution ψ_h for the cubic triangle and 1-point integration of the penalty term converges to ψ in the norm $|||\cdot|||_h$ as

$$|||\psi - \psi_h|||_h \leq C h^\mu \|\psi\|_{r,\Omega} \tag{11}$$

where

$$\mu = \tfrac{1}{2}\min(\sigma - 1, 2r - 3 - \sigma, 5 - \sigma) \tag{12}$$

and we have used $\varepsilon = Ch^\sigma$, constant C.

Proof. The proof follows as an extension of the analysis of Babuška and Zlamal[12] to the case of reduced integration. (See also References 13 and 16 for details.) ∎

Remark

Note that the best rate $O(h)$ in this norm is achieved for $\sigma = r - 1$ if $r \leq 4$. For smooth solutions ($r \geq 4$) the best rate is obtained for $\sigma = 3$. That is, $\varepsilon = Ch^3$.

Corollary

The result (11), (12) applies in the norm $\|\cdot\|_h$ given by

$$\|v\|_h^2 = \sum_{e=1}^{E} \int_{\Omega_e} \nu(\Delta v)^2 \mathrm{d}x\mathrm{d}y + \sum_{s=1}^{S} \frac{1}{2\varepsilon} \int_{\Gamma_s} \left[\frac{\partial v}{\partial n}\right]^2 \mathrm{d}s \tag{13}$$

Proof. We first note that

$$\|v\|_h^2 - |||v|||_h^2 = \frac{1}{2\varepsilon}\left\{\sum_{s=1}^{S}\left[\int_{\Gamma_s}\left[\frac{\partial v}{\partial n}\right]^2 \mathrm{d}s - I_s\left(\left[\frac{\partial v}{\partial n}\right]^2\right)\right]\right\} \tag{14}$$

Setting $e = \psi - \psi_h$ for v in (14) and using $[\![\partial e/\partial n]\!] = [\![\partial \psi_h/\partial n]\!]$

$$\|e\|_h^2 - |||e|||_h^2 = \frac{1}{2\varepsilon}\left\{\sum_{s=1}^{S}\left[\int_{\Gamma_S}\left[\!\left[\frac{\partial \psi_h}{\partial n}\right]\!\right]^2 - I_s\left(\left[\!\left[\frac{\partial \psi_h}{\partial n}\right]\!\right]^2\right)\right]\right\} \tag{15}$$

In particular, since $[\![\partial \psi_h/\partial n]\!]$ is quadratic on Γ_S and zero at the end points it achieves its maximum magnitude at the central Gauss point. Hence $[\![\partial \psi_h/\partial n]\!]^2 > 0$ on Γ_S and achieves its maximum at the mid-point. It follows that for the 1-point rule $I_s(\cdot)$ in (15),

$$\|e\|_h^2 - |||e|||_h^2 \leqslant 0$$

so that

$$\|e\|_h \leqslant |||e|||_h \tag{16}$$

which implies that the estimate holds in the new norm.

Remarks:

1. The estimates also hold in the norm given by
$$|v|_h^2 = \sum_{e=1}^{E}\int_{\Omega_e} v\Delta v\Delta v \mathrm{d}x\mathrm{d}y$$
introduced by Ciarlet[17] for non-conforming methods.
2. The estimates can be shown to hold in the H^1 norm using the approach of Babuška and Zlamal.[12]
3. We emphasize that the estimates are for the penalty method with 1-point quadrature and that the extensions here refer only to the norm in which the error in this solution is measured.

NUMERICAL EXPERIMENTS

As a model problem with smooth solution satisfying the conditions of the theorem we take the example introduced by Johnson and Pitkaranta[14] for studying the primitive variable approximations. The flow problem for ψ is given by

$$\Delta^2\psi = f/v, \quad \text{in } (0,1)\times(0,1)$$

with

$$\psi = \frac{\partial \psi}{\partial n} = 0, \quad \text{on } \partial\Omega \tag{17}$$

where f/v is calculated to correspond to the smooth stream function

$$\psi(x, y) = x^2y^2(1-x)^2(1-y)^2 \tag{18}$$

In the following numerical studies the square domain is discretized to a uniform mesh of right isosceles triangles as shown in Figure 1 and having a characteristic mesh length of size h along horizontal and vertical sides. The behaviour of the error in the H^1-norm as the mesh is refined for the non-conforming method for this problem is shown in Figure 2. The numerical solution to the penalty problem is computed on successive uniform meshes with $h = 1/2$, $1/4$, $1/6$, $1/8$, $1/10$ and $1/16$ for $\varepsilon = Ch^\sigma$, $\sigma = 3$ and penalty constants $C = 1$, 10^{-2} and 10^{-4}. In the first set of calculations *exact* integration (3-point rule) of the penalty term is used. The results are shown in Figure 3. This numerical study reveals the notable result that the method fails to converge for these values, the theoretical estimates notwithstanding. We note further (Figure 4) that the solutions on successively

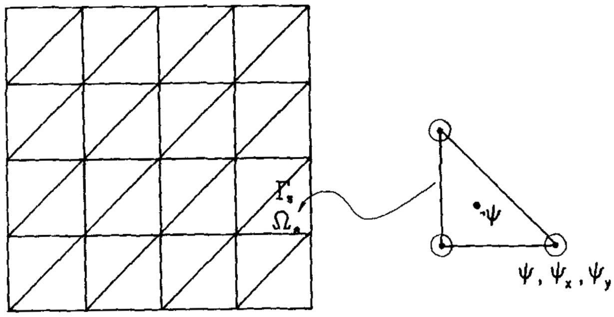

Figure 1. Sample discretization of domain by Hermite triangles for convergence studies showing typical element Ω_e and interface Γ_S along which penalty constraint applies

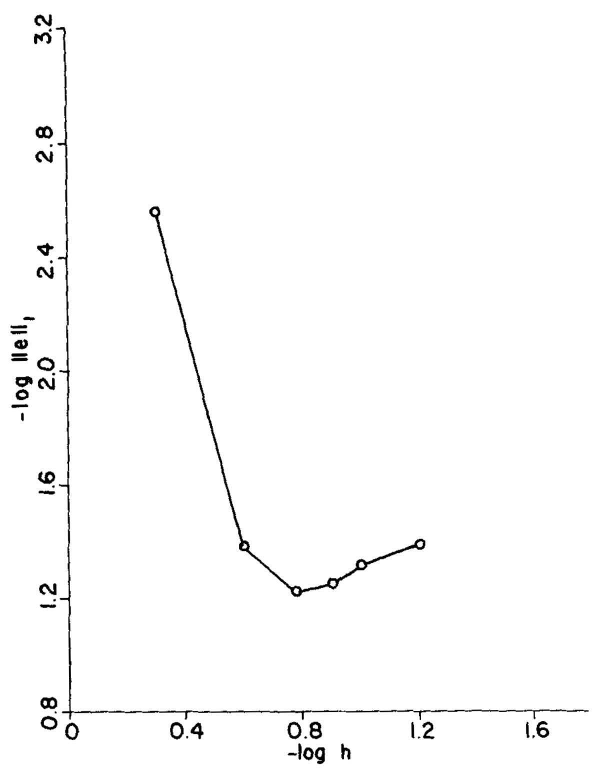

Figure 2. Global error in the H^1-norm for the non-conforming method

refined meshes converge towards $\psi_h = 0$ which might be interpreted as a 'locking' behaviour similar to that observed in some other penalty and mixed finite element methods.

Motivated by similar considerations in other penalty finite element studies we examine the effect of reduced integration. For the 2-point rule the method again fails in the same way as for exact integration. However, with the 1-point rule we obtain a convergent method as indicated by the

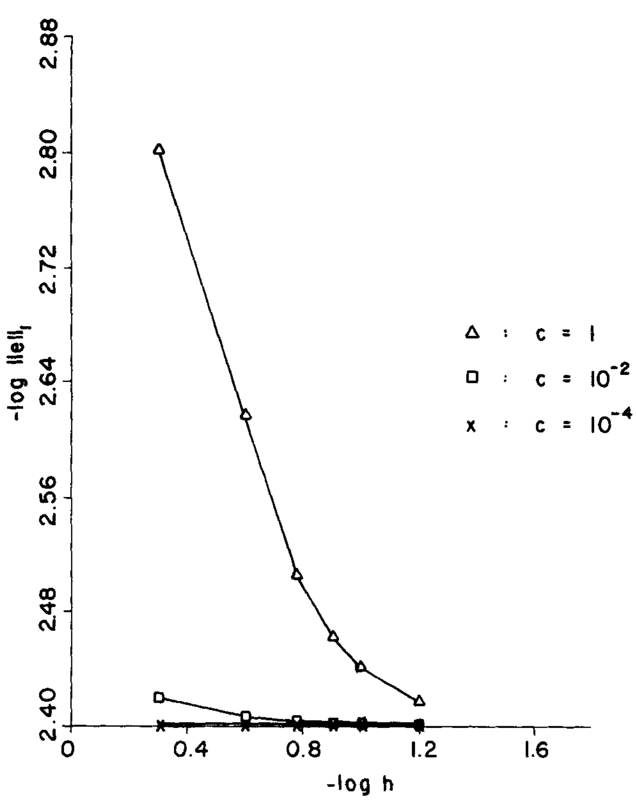

Figure 3. Global errors for the solution in the H^1-norm for different values of penalty constant C, and using exact integration of penalty term

results in Figure 5. Larger values of C (e.g. $C = 10^2$) produce results that are poor on coarse meshes as the penalty is not sufficiently strongly enforced, but converge as the mesh is further refined. The rates in the norms $|||\cdot|||_h$ and $\|\cdot\|_1$ are given by the slopes in Figure 6.

INTERPRETATION FROM RANK CONDITIONS

The failure of the exact and 2-point quadrature schemes can be interpreted using the equivalence of the penalty method to a multiplier method. In the present instance the multiplier method is a hybrid method with the approximate multiplier determined from the finite element solution ψ_h^ε such that

$$\lambda_h^\varepsilon(\xi_i) = \frac{1}{\varepsilon}\left[\!\left[\frac{\partial \psi_h^\varepsilon(\xi_i)}{\partial n}\right]\!\right], \quad 1 \leqslant i \leqslant G \tag{19}$$

at each Gauss point ξ_i on Γ_S. For the 1-, 2- and 3-point rules the spaces of multipliers Λ^h are piecewise constant, linear and quadratic, respectively. The rank condition is derived from the inf-sup or stability condition for a saddle-point problem. It may be expressed formally in the following way:

$$I\left(\mu_h\left[\!\left[\frac{\partial v_h}{\partial n}\right]\!\right]\right) = 0, \quad \text{for all } v_h \in H^h \tag{20}$$

implies that $\mu_h = 0$.

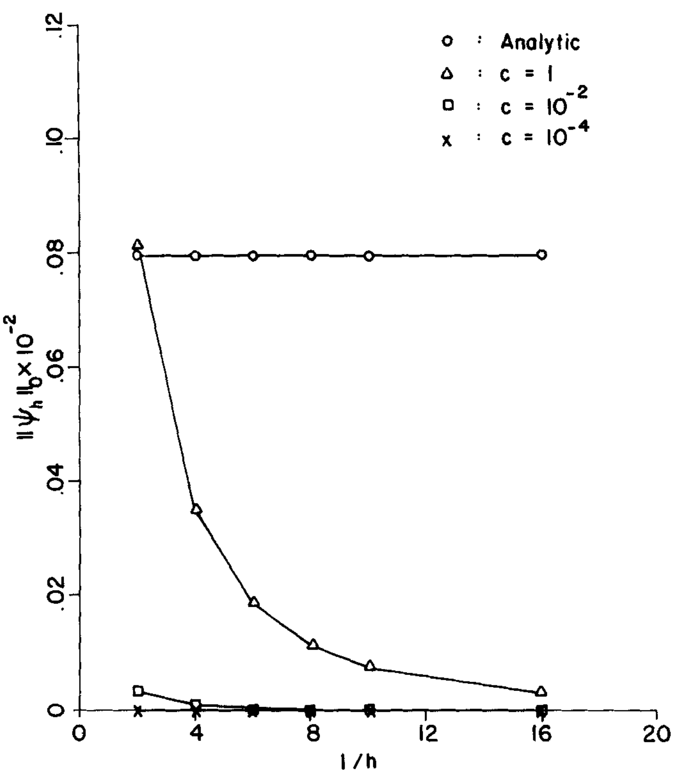

Figure 4. L^2-norm of the solution and of its approximation obtained using exact integration of penalty terms

If $\mu_h \in P_t(\partial\Omega_e)$ and $[\![\partial v_h/\partial n]\!] \in P_k(\partial\Omega_e)$ for triangular elements, where P_s denotes the class of polynomials of degree s, then (20) implies $\mu_h = 0$ if and only if*

$$\begin{aligned} t &\leqslant k-1, \quad \text{if } k \text{ is odd} \\ t &\leqslant k-2, \quad \text{if } k \text{ is even} \end{aligned} \tag{21}$$

In the present case $[\![\partial v_h/\partial n]\!]$ is quadratic ($k=2$) and we find that (21) holds for $t=0$ (piecewise constant or 1-point rule) but fails for $t=1,2$ (piecewise linear and quadratic multipliers defined by the 2-point and 3-point rules).

DRIVEN CAVITY CALCULATION

As a more practical example we consider the well-known driven cavity problem. This example is frequently used as a test case for finite difference and finite element programs because of the abundance of other computed solutions. In other respects, however, we remark that this case is far from ideal as we have no analytic solution and singularities are present in the boundary data at the corners adjacent to the driven surface. It does, however, raise some additional interesting points concerning our penalty method.

*We have recently generalized the results given by Carey and Oden[15] for second-order problems to fourth-order problems to obtain this result.[16]

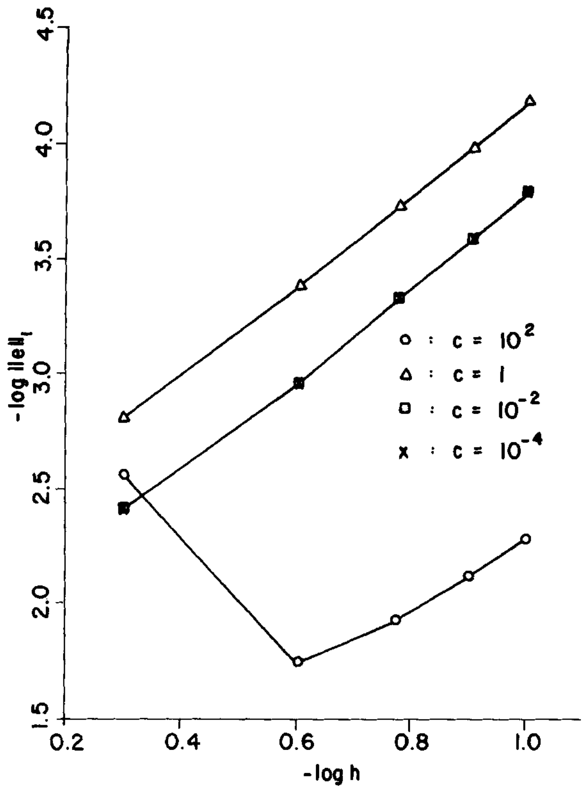

Figure 5. Global rates of convergence for the solution in the H^1-norm for different values of penalty constant C and 1-point Gauss rule

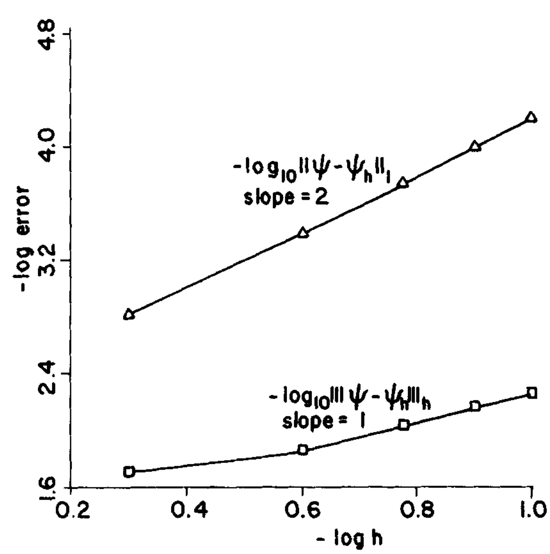

Figure 6. Global rates of convergence for the solution in the $\|\cdot\|_1$ and $|||\cdot|||_h$ norms for $C = 1$ and 1-point rule

There are no body forces and the governing equation becomes

$$\Delta^2\psi = 0, \quad \text{in } \Omega = (0,1) \times (0,1) \tag{22}$$

with

$$\psi = 0, \quad \text{on } \partial\Omega \tag{23}$$

and from the 'no-slip' requirement

$$\psi_x = 0, \quad \text{on } \partial\Omega$$
$$\psi_y = 0, \quad \text{on } y = 0, x = 0 \text{ and } x = 1 \tag{24}$$

$\psi_y = 1$, on $y = 1$ (the driven surface)

The corresponding variational problem is solved using the Hermite cubic element with inter-element penalty treated using the 1-point Gauss rule. Streamline contours are plotted in Figure 7 for solution on a uniform mesh with $h = 1/10$. The velocity components along the lines $x = 1/2$ and $y = 1/2$ are plotted in Figure 8. These results compare well with other published results for this problem.

In closing we make some interesting observations concerning the treatment of boundary data

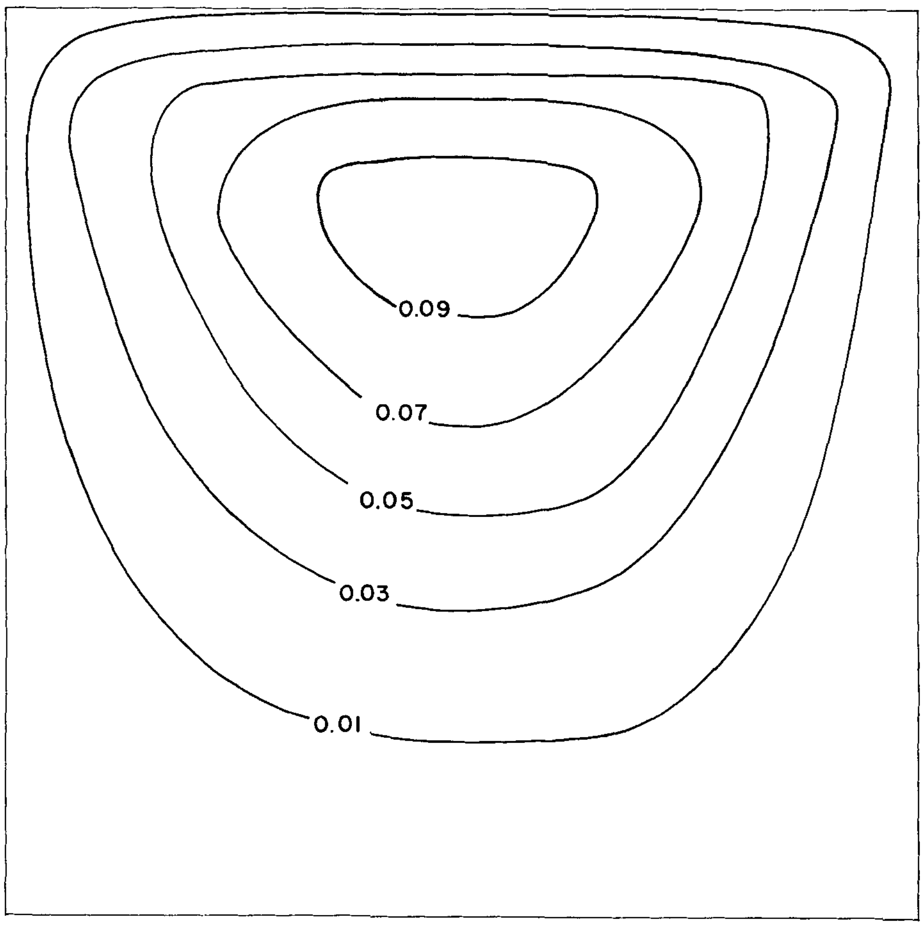

Figure 7. Contour plot of stream function for driven cavity

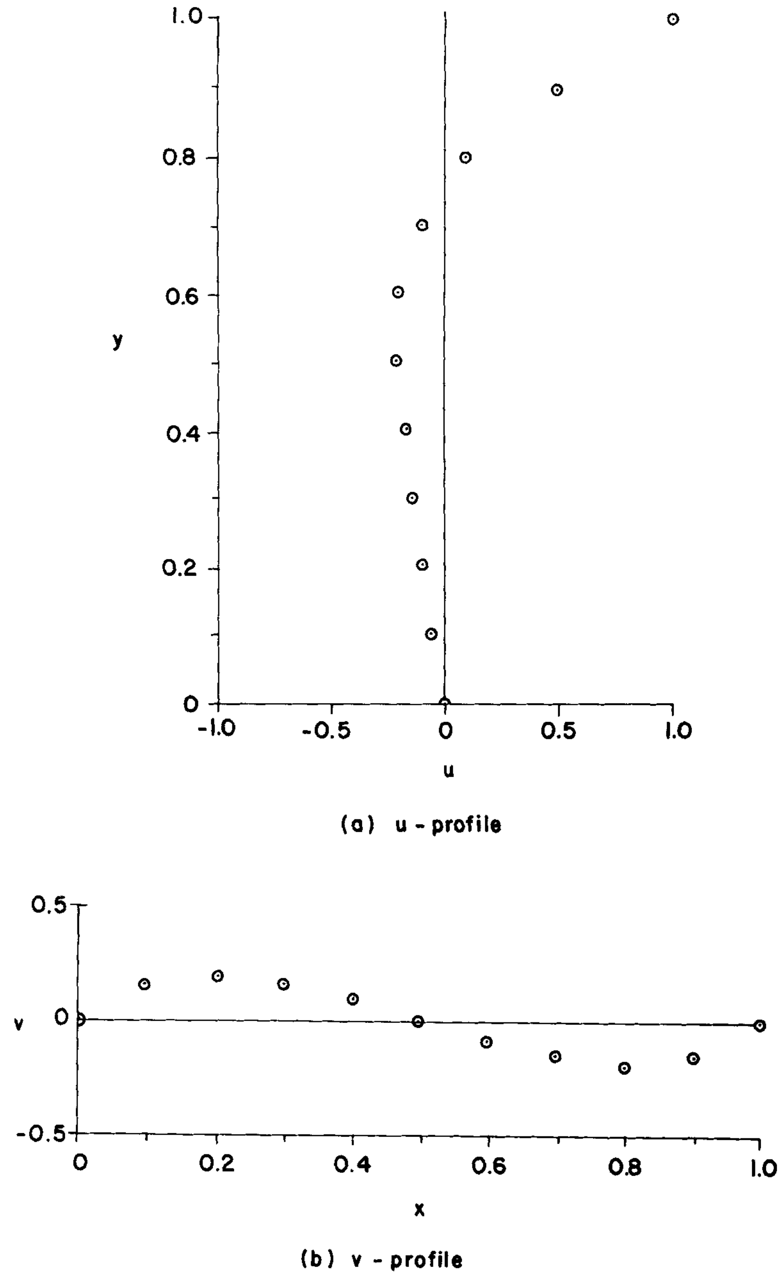

Figure 8. Velocity profiles along the lines $x = 1/2$ and $y = 1/2$: (a) u-profile; (b) v-profile

and effect of the corner singularities. In these calculations the data (23), (24) are specified as interpolated point values on $\partial\Omega$ with $\psi_y = 1$ at the corner points (0, 1) and (1, 1). Values of ψ, ψ_x and ψ_y are interpolated at the boundary nodes. This implies that $\psi = 0$ on $\partial\Omega$ is satisfied identically but the normal derivative $\psi_n = 0$ or 1 (as appropriate) holds only at the nodes. That is, boundary conditions involving the normal to a boundary side only are satisfied at the boundary nodes. We can strengthen this by adding a penalty to enforce $\psi_y - 1 = 0$ along $y = 1$ to produce results closely agreeing with those in Figures 7 and 8. Adding a further boundary penalty on the remaining sides yields results very different from those in Figures 7 and 8. In fact such a boundary penalty on all sides produces velocities with magnitude greater than unity at some points in the interior.

A possible interpretation of this can be conjectured as follows: The standard methods for approximate solution of this problem relax the conditions at the upper corners so that the velocity in the finite element problem changes continuously. Physically this permits inflow or outflow of fluid through the boundary sides of the elements adjacent to these corners. When the boundary penalty is used to enforce the given conditions the jump in the velocity at the corners is more strongly felt since the penalty acts to enforce the condition along the remainder of the side and the approximation must then rise abruptly to satisfy $\psi_y = 1$ at the upper corners. The abrupt change in our coarse mesh solution leads to significant oscillatory perturbations in the interior. Finally, to corroborate this interpretation we applied the penalty on all of the boundary except the vertical element sides at the corners. The resulting solution closely agrees with that in Figures 7 and 8.

CONCLUDING REMARKS

Inter-element penalties have been introduced as a technique for correcting the non-conforming Hermite cubic triangle. The method is stable and hence convergent if the penalty term is computed using 1-point Gauss integration. For a model problem with smooth solution the penalty $\varepsilon = Ch^{\sigma}$ produces best rates of convergence for $\sigma = 3$ and the rate is confirmed by numerical experiment. A final computation considers the driven cavity problem and shows that the method yields accurate results. The effect of the corner singularities in the presence of a boundary penalty proceduces some interesting results. We remark in closing that clearly the element and penalty procedure used here can be applied equally to other fourth-order problems such as those of plate bending.

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